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PROJECTION METHOD OF SOLVING A PROBLEM OF FORCED UNSTEADY OSCILLATIONS IN NONLINEAR SYSTEMS WITH SLOWLY VARYING PARAMETERS*

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A problem of forced unsteady oscillations of a nonlinear system where its parameters, as well as the parameters of the external periodic perturbation vary slowly, is considered. It is assumed that the nonstationary oscillation process is preceded by the stationary periodic oscillations. A projection method is proposed, which makes it possible to construct solutions with any degree of accuracy after a finite time interval.

1. Formulation of the problem and description of the method. The problem is described by the following equation:

$$x^{\cdot}(t) = F(x(t), \tau(t), \varphi(t)) \tag{1.1}$$

where x is an m-dimensional vector, F is a vector function 2π-periodic in φ, and τ, called "slow time", is a known function of the "rapid time" t satisfying the relations

$$\tau^{\cdot} = \varepsilon g(\tau),\ t \geqslant 0;\ \tau(t) = 0,\ t \leqslant 0 \tag{1.2}$$

where ε is a small nonnegative number, $\varphi^{\cdot}(t) = \omega(\tau) > 0$ is the perturbation frequency, $\varphi(0) = 0$, the functions $g(\tau)$ and $\omega(\tau)$ are continuous and $g(\tau) > 0$).

According to the condition (1.2) the parameters of the system and the perturbation frequency $\omega(0) = \omega^{\circ}$ are all constant when $t \leqslant 0$. We also assume that the system executes purely periodic motions $x(t) = \xi^{\circ}(\omega^{\circ}t)$ with the frequency ω°, and this determines the initial condition

$$x(0) = \xi^{\circ}(0) \tag{1.3}$$

The problem is investigated under the assumption that the function F is defined and condinuously differentiable for x belonging to the open domain Ω and for any τ and φ. We also assume that its partial derivatives satisfy the Lipschitz condition in x in any bounded region of the domain of definition.

Substituting $t_1 = \varphi(t)$ by an independent variable reduces the equation (1.1) to the form

$$\frac{dx}{dt_1} = F_1(x, \tau(t_1), t_1);\quad F_1 = \frac{F}{\omega},\quad \frac{d\tau}{dt_1} = \varepsilon\frac{g(\tau)}{\omega(\tau)} = \varepsilon g_1(\tau)$$

or, neglecting the indices accompanying t, g and F, to

$$x^{\cdot} = F(x, \tau(t), t) \tag{1.4}$$

When $t \leqslant 0$ and $\varepsilon > 0$, and also when t is arbitrary and $\varepsilon = 0$, the system (1.4) becomes a 2π-periodic system

$$x^{\cdot} = F(x, 0, t) \tag{1.5}$$

which has a 2π-periodic solution $\xi^{\circ}(t)$ by definition.

An approximate solution of the problem (1.5) of steady state oscillations can be found using the projection method, in the form

$$\xi_n^{\circ}(t) = \sum_{k=-n}^{n} \alpha_{kn}^{\circ} e^{ikt} \tag{1.6}$$

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The constants α_{kn}° $(-n\leqslant k\leqslant n)$ are obtained from the system of nonlinear Galerkin equations (n denotes the number of approximation (see e.g. /1/))

$$F_{kn}(\alpha_{-nn}^\circ,\ \ldots,\ \alpha_{nn}^\circ,\ 0)=0,\quad |k|\leqslant n \tag{1.7}$$

$$F_{kn}(\alpha_{-nn},\ldots,\alpha_{nn},\tau)=\int\limits_0^{2\pi} F\left(\sum_{l=-n}^{n}\alpha_{ln}e^{ilt},\ \tau,\ t\right)e^{-ikt}\frac{dt}{2\pi}-ik\alpha_{kn} \tag{1.8}$$

It is understood that, when real solutions of a real system are sought, then $\alpha_{-kn}=\bar{\alpha}_{kn}$.

To obtain an approximate solution of the problem (1.4), (1.3) of nonsteady oscillations, we propose a method according to which the solutions are obtained in the form

$$x_n(t)=\sum_{k=-n}^{n}\alpha_{kn}(t)\,e^{ikt} \tag{1.9}$$

and the coefficients α_{kn} are found from a system with initial conditions

$$\alpha_{kn}^{\cdot}=F_{kn}(\alpha_{-nn},\ \ldots,\ \alpha_{nn},\ \tau(t)),\quad |k|\leqslant n \tag{1.10}$$

$$\alpha_{kn}(0)=\alpha_{kn}^\circ \tag{1.11}$$

with F_{kn} taken from (1.8).

When the relevant assumptions ensuring the asymptotic stability of the steady mode of oscillations, the coefficients $\alpha_{kn}(t)$ are found to be slow functions of time. For this reason the Cauchy problem (1.10), (1.11) can be solved using one of the numerical step methods.

If real solutions are sought, then it is convenient to introduce new (real) unknowns a_{kn} $(0\leqslant k\leqslant n)$ and b_{kn} $(1\leqslant k\leqslant n)$, or r_{kn} $(0\leqslant k\leqslant n)$ and ψ_{kn} $(1\leqslant k\leqslant n)$ defined by the equations

$$2\alpha_{kn}=a_{kn}-ib_{kn}=r_{kn}\exp(-i\psi_{kn})\quad (0\leqslant k\leqslant n,\ b_{0n}=\psi_{0n}=0)$$

In this case the relations (1.9) and (1.10) yield

$$x_n(t)=\frac{1}{2}a_{0n}(t)+\sum_{k=1}^{n}(a_{kn}(t)\cos kt+b_{kn}(t)\sin kt)=\frac{1}{2}r_{0n}(t)+\sum_{k=1}^{n}r_{kn}(t)\cos(kt-\psi_{kn})$$

$$a_{kn}^{\cdot}=\frac{1}{\pi}\int\limits_0^{2\pi}F\left(\frac{1}{2}a_{0n}+\sum_{l=1}^{n}(a_{ln}\cos l\varphi+b_{ln}\sin l\varphi),\ \tau(t),\ \varphi\right)\cos k\varphi\, d\varphi-kb_{kn} \tag{1.12}$$

$$b_{kn}^{\cdot}=\frac{1}{\pi}\int\limits_0^{2\pi}F\left(\frac{1}{2}a_{0n}+\sum_{l=1}^{n}(a_{ln}\cos l\varphi+b_{ln}\sin l\varphi),\ \tau(t),\ \varphi\right)\sin k\varphi\, d\varphi+ka_{kn} \tag{1.13}$$

$$r_{kn}^{\cdot}=\frac{1}{\pi}\int\limits_0^{2\pi}F\left(\frac{1}{2}r_{0n}+\sum_{l=1}^{n}r_{ln}\cos(l\varphi-\psi_{ln}),\ \tau(t),\ \varphi\right)\cos(\psi_{kn}-k\varphi)\,d\varphi \tag{1.14}$$

$$\psi_{kn}^{\cdot}=k-\frac{1}{\pi r_{kn}}\int\limits_0^{2\pi}F\left(\frac{1}{2}r_{0n}+\sum_{l=1}^{n}r_{ln}(\cos(l\varphi-\psi_{ln}),\ \tau(t),\ \varphi\right)\sin(\psi_{kn}-k\varphi)\,d\varphi \tag{1.15}$$

When the initial system (1.4) is linear or quasi-linear, so is the system (1.12), (1.13). The system (1.14), (1.15) is always nonlinear, but has the advantage that it allows a direct determination of the amplitudes and phases determining the character of the oscillations. The method described can be regarded as a projection method, provided that we pass to a family of the problems analogous to (1.4), (1.3) and depending on the parameter.

2. Projection approach to the method. The family of equations

$$x^{\cdot}(t,\psi)=F(x(t,\psi),\tau(t),\ t+\psi) \tag{2.1}$$

depending on the parameter ψ has, at $t\leqslant 0$, a family of periodic solutions $\xi^\circ(t+\psi)$. Their continuation at $t\geqslant 0$ is found from the initial conditions

$$x(0,\psi)=\xi^\circ(\psi) \tag{2.2}$$

The Cauchy problem (2.1), (2.2) has the corresponding abridged problem

$$x_n^{\cdot}(t,\psi)=P_nF(x_n(t,\psi),\tau(t),\ t+\psi),\quad x_n(0,\psi)=\xi_n^\circ(\psi) \tag{2.3}$$

where P_n is the Fourier operator forming, together with the 2π-periodic function of the argument ψ, its n-th Fourier sum, and ξ_n° is the approximation (1.6) of the stationary problem

$$x_n(t,\psi)=\sum_{k=-n}^{n}\alpha_{kn}(t)\,e^{ik(t+\psi)}=\sum_{k=-n}^{n}\beta_{kn}(t)\,e^{ik\psi} \tag{2.4}$$

The problem (2.3) is equivalent to the system (1.10), (1.11) for the coefficients $\alpha_{kn}(t)$, therefore for the approximation to the initial problem (1.4), (1.3) we have $x_n(t) = x_n(t, 0)$. Assuming $\Phi(x, t, \psi) = F(x, \tau(t), t+\psi)$, we can write the problem (2.1), (2.2) and the problem (2.3) in the general form

$$x^{\cdot}(t,\psi)=\Phi(x(t,\psi),t,\psi),\ x(0,\psi)=\xi^\circ(\psi) \tag{2.5}$$

$$x_n^{\cdot}(t,\psi)=P_n\Phi(x_n(t,\psi),t,\psi),\ x_n(0,\psi)=\xi_n^\circ(\psi) \tag{2.6}$$

The above problem are studied under the assumption that the function Φ is continuous and has continuous partial derivatives in x and ψ satisfying the Lipschitz condition in x in any bounded region of the domain of definition, which is guaranteed by the properties of F.

Equations of the problems (2.5) and (2.6) can be interpreted as equations in the Banach space of functions of the parameter ψ. Let $E = W_{1,2}^m$ be a Sobolev space of m-dimensional, 2π-periodic vector functions $\xi(\psi)$, absolutely continuous and possessing a square-summable derivative almost everywhere. The norm is given by the equation

$$\|\xi\|^2_{W^m_{1,2}}=\|\xi\|^2_{L^m_2}+\left\|\frac{d\xi}{d\psi}\right\|^2_{L^m_2}=\int_0^{2\pi}\left(|\xi(\psi)|^2+\left|\frac{d\xi}{d\psi}\right|^2\right)d\psi \tag{2.7}$$

By virtue of the inclusion theorem /2/ this norm is stronger than the norm of uniform convergence

$$\|\xi\|_C=\max_{0\leqslant\psi\leqslant2\pi}|\xi(\psi)|\leqslant K\|\xi\|_{W^m_{1,2}} \tag{2.8}$$

Let H be an open set of functions belonging to E, the values of which lie in Ω. We define the function $\eta(\psi)=\Phi(\xi(\psi), t, \psi)$ for any ξ belonging to H and any t. From (2.8) and the assumption made about the properties of Φ, it follows that $\eta = f(\xi, t)$ is a function with the values belonging to E, defined for ξ belonging to H and for all t. Moreover, $f(\xi, t)$ is a continuous function of its arguments satisfying the Lipschitz condition in ξ in every bounded region of the domain of definition.

Introducing the time functions with values belonging to H

$$\xi(t)(\psi)=x(t,\psi),\quad \xi_n(t)(\psi)=x_n(t,\psi) \tag{2.9}$$

we find that the problems (2.5) and (2.6) have the corresponding Cauchy problems in H

$$\xi^{\cdot}(t)=f(\xi(t),t),\quad \xi(0)=\xi^\circ \tag{2.10}$$

$$\xi^{\cdot}_n(t)=P_nf(\xi(t),t),\ \xi_n(0)=\xi_n^\circ \tag{2.11}$$

Using the theorem of existence and uniqueness for the Cauchy problem in a Banach space /3/, we can prove the following assertion.

Lemma 1. Problems (2.10) and (2.11) are equivalent to the problems (2.5) and (2.6) i.e. their solutions are defined (and unique) on one and the same time interval, and are connected by the relation (2.9).

3. Convergence of the method. Let E be any Banach space and $f(\xi, t)$ a continuous function defined on its open subset H and time interval $J = [0, b)$, assuming the values in E and satisfying the Lipschitz condition in ξ in any bounded region of the domain of definition. Let also $P_n : E \to E_n$ be a sequence of linear bounded operators mapping E onto a closed invariant subsapce $(P_nE_n \subset E_n)$. We introduce the following notation

$$\|\xi(t)\|_T=\max_{0\leqslant t\leqslant T}\|\xi(t)\|$$

Theorem 1. Let $\xi(t)$ be a solution of the problem (2.10) on the (finite or infinite) interval $J = [0, b)$, and let

$$P_n\xi \to \xi, \ \forall \xi \text{ from } E \tag{3.1}$$

$$\xi_n^\circ \to \xi^\circ \tag{3.2}$$

as $n \to \infty$. Then a solution $\xi_n(t)$ of the problem (2.11) exists on any finite interval $[0, T] \subset J$ for all sufficiently large n, and $\xi_n(t) \to \xi(t)$ as $n \to \infty$. The rate of convergence is determined by the inequality (c_1 and c_2 are constants)

$$\| \xi_n(t) - \xi(t) \|_T \leqslant \| \xi_n^\circ - P_n\xi^\circ \| c_1 + \| P_n\xi(t) - \xi(t) \|_T c_2 \tag{3.3}$$

Proof. Let $r > 0$ be such that the r-neighborhood of $\xi(t)$ remains in H for $0 \leqslant t \leqslant T$. By virtue of (3.1) $\| P_n \| \leqslant L$ and $\| P_n\xi(t) - \xi(t) \|_T \to 0$, hence at large n the function $P_n\xi(t)$ remains in the region $H_M = \{\xi \in H \mid \| \xi \| < M\}$ where $M = \| \xi(t) \|_T + r$. From this it follows that $P_n\xi(t)$ represents the ε_n-solution of (2.11) for $0 \leqslant t \leqslant T$ where $\varepsilon_n = LK \| \xi \cdot(t) - P_n\xi(t) \|$, and K is the Lipschitz constant of $f(\xi, t)$ in H_M for $0 \leqslant t \leqslant T$. Using the theorem on matching the ε-solutions /3/, we can show that at large n the function $\xi_n(t)$ is defined and does not emerge from H_M onto $[0, T]$, and thus derive the estimate from above for $\| P_n\xi(t) - \xi_n(t) \|$ from which the estimate (3.3) follows.

Theorem 2. Let the right-hand part of (2.5) satisfy the condition of smoothness formulated earlier, and $\xi_n^\circ \to \xi^\circ$ on the norm (2.7). The the solution $x_n(t, \psi)$ of the problem (2.6) converges to the solution $x(t, \psi)$ of (2.5), in the sense that for any $T > 0$ and $n \to \infty$

$$\max_{0 \leqslant t \leqslant T} \left[\int_0^{2\pi} | x_n(t, \psi) - x(t, \psi) |^2 d\psi + \int_0^{2\pi} \left| \frac{\partial x_n(t, \psi)}{\partial \psi} - \frac{\partial x(t, \psi)}{\partial \psi} \right|^2 d\psi \right] \to 0 \tag{3.4}$$

The rate of convergence is determined by the rate of convergence of $\xi_n^\circ \to \xi^\circ$ and of the Fourier series for $x(t, \psi)$ and $\partial x(t, \psi)/\partial \psi$.

Proof. By virtue of Lemma 1, it is sufficient to confirm the validity of condition (3.1) of Theorem 1 for $E = W_{1,2}^m$ and the Fourier operators P_n. But this is equivalent to the convergence of the Fourier series in L_2^m for ξ and $d\xi/d\psi$ for $\xi \in W_{1,2}^m$.

Corollaries. 1°. When the conditions of Theorem 2 hold, we have the uniform convergence $x_n(t, \psi) \to x(t, \psi)$, i.e.

$$\max_{\substack{0 \leqslant t \leqslant T \\ 0 \leqslant \psi \leqslant 2\pi}} | x_n(t, \psi) - x(t, \psi) | \to 0, \quad n \to \infty \tag{3.5}$$

2°. When the conditions of Theorem 2 hold, we have the uniform convergence of the solution $x_n(t)$ obtained from (1.9), (1.10) and (1.11) to the solution $x(t)$ of the problem (1.3), (1.4), i.e.

$$\max_{0 \leqslant t \leqslant T} | x_n(t) - x(t) | \to 0, \quad n \to \infty \tag{3.6}$$

The first corollary follows from Theorem 2 by virtue of the inequality (2.8), and the second corollary follows from the first one since $x_n(t) = x_n(t, 0)$, $x(t) = x(t, 0)$.

The requirement that the right-hand side of (1.1) be smooth, can be relaxed by using a modification of the proposed method in which the Fourier operators P_n are replaced by the Fejér operators forming, together with the 2π-periodic function $\xi(\psi)$, its Fejér sum (α_k are the Fourier coefficients)

$$\sum_{k=-n}^{n} (1 - k/(n+1)) \alpha_k e^{ik\psi}$$

If in addition the right-hand part of the system (1.1) is continuous and satisfies the Lipschitz condition in x on the bounded sets, then, taking the space of continuous functions of ψ as E, we can confirm the validity of Lemma 1. Therefore Theorem 1 leads to an assertion analogous to Theorem 2 where (3.4) is replaced by (3.5) only, with (3.6) following as a corollary.

4. Slowness of the change in the coefficients $\alpha_{kn}(t)$. At small ε the coefficients $\alpha_{kn}(t)$ are slow functions of time t on any finite interval. Indeed, the continuity of the solutions with respect to the parameter ε implies that when $\varepsilon \to 0$, $\alpha_{kn}(t) \to \alpha_{kn}^{\circ}(t)$ uniformly in $t \in [0, T]$ for any $T > 0$. Therefore $\alpha_{kn}^{\cdot}(t) \to F_{kn}(\alpha_{-nn}^{\circ}, \ldots, \alpha_{nn}^{\circ}, 0) = 0$. In a number of cases however (e.g. during a passage through a resonance), it is necessary to consider the solution on a finite interval of slow time τ, i.e. on an interval of time t asymptotically unbounded when $\varepsilon \to 0$.

Let us assume (for simplicity, but without loss of generality) that $\tau = \varepsilon t$ when $\varepsilon \geqslant 0$. Assuming $\alpha_n = (\alpha_{-nn}, \ldots, \alpha_{nn})$, we can write the equations (1.4) and (1.10) in the form

$$\varepsilon \frac{dx}{d\tau} = F(x, \tau, t), \quad \varepsilon \frac{d\alpha_n}{d\tau} = F_n(\alpha_n, \tau) \tag{4.1}$$

Let the equation $F_n(\alpha_n, \tau^*) = 0$ have an isolated root $\alpha_n^*(\tau^*)$ for $0 \leqslant \tau^* \leqslant T$, which is an asymptotically stable uniformly in τ^*, stationary solution of the equation

$$d\alpha_n / dt = F_n(\alpha_n, \tau^*) \tag{4.2}$$

Then by virtue of the Tikhonov theorem /5/ the solution $\alpha_n(t)$ of the second system (4.1) tends to $\gamma_n(t) = \alpha_n^*(\varepsilon t) = \alpha_n^*(\tau)$ as $\varepsilon \to 0$ uniformly in τ for $0 < t_1 \leqslant \tau \leqslant T$. Moreover, if the solution of the system (1.7) is used as the initial condition α_n° for (4.2) so that $\alpha_n^*(0) = \alpha_n(0)$, then the convergence will be uniform over the whole interval $0 \leqslant \tau \leqslant T$. Thus $\alpha_n(t) = \gamma_n(t) + \delta_n(t)$ where $d\gamma_n/dt$, $\delta_n \to 0$ as $\varepsilon \to 0$ uniformly in τ for $0 \leqslant \tau \leqslant T$. In this case one can speak of asymptotic slowness of the solution $\alpha_n(t)$.

The conditions of the Tikhonov theorem can be verified directly using the autonomous equation (4.2). However the derivation of these conditions from the properties of the equation

$$dx/dt = F(x, \tau^*, t) \tag{4.3}$$

obtained from (1.4) by "freezing" τ ($\tau^* = \mathrm{const}$) can be of interest.

Theorem 3. Let the equation (4.3) have, for $0 \leqslant \tau^* \leqslant T$, a periodic solution $x(\tau^*, t)$ depending continuously on τ^*, and let all multiplying factors of the equation in variations lie within a unit circle when $0 \leqslant \tau^* \leqslant T$. Then, for sufficiently large n, the solution $\alpha_n(t)$ of the second equation of (4.1) with $\alpha_n(0) = \alpha_n^*(0)$ will be asymptotically slow as $\varepsilon \to 0$.

Proof. Equation (4.3) can be replaced by an equivalent integral equation of the form $x = Bx$ in the space $E = C(W_{1,2}^m)$ of continuous functions of τ^* with the values belonging to the space $W_{1,2}^m$ of functions of t. The equation $F_n(\alpha_n, \tau^*) = 0$ is equivalent to the Galerkin equation $x_n = P_n B x_n$, therefore from Theorem 19.1 of /4/ it follows that for large n there exists an isolated stationary solution $\alpha_n^*(\tau^*)$ of equation (4.2) continuous in τ^*. To prove its asymptotic stability uniform in τ^*, we shall consider the equation $x^{\cdot} = F(x, \tau^*, t + \psi)$ and its solution $x(\tau^*, t + \psi)$ periodic in t and ψ continuous in τ^*. It can be shown that the monodromy operator $U(\tau^*, \psi)$ of the corresponding equation in variations resembles, at any ψ, the operator $U(\tau^*, 0)$, and this means that its spectrum lies within a unit circle. On the other hand, Theorem 1 implies that the monodromy operator $U_n(\tau^*, \psi)$ of the equation $x_n^{\cdot} = P_n F(x_n, \tau^*, t + \psi)$ corresponding to the solution $x_n(\tau^*, t + \psi)$ converges to the operator $U(\tau^*, \psi)$ in the space $E = C(L_2^{m \times m})$ of operator functions continuous in τ^* and square-summable in ψ. From this it follows that its spectrum lies, at large n, within a unit circle. Therefore the solution $x_n(\tau^*, t + \psi)$, and hence $\alpha_n^*(\tau^*)$, is asymptotically stable uniformly in τ^*.

Under the stricter conditions of stability of the solution $x(\tau^*, t)$, we can guarantee the slowness of all $\alpha_n(t)$ (for which $\alpha_n^*(\tau^*)$ exists) and not only of those with sufficiently large n. Below we use the notation $\operatorname{Re} B = (B + B^*)/2$ where B^* is a conjugate of B.

Theorem 4. Let the operator $\operatorname{Re}[W \partial F \cdot (x, \tau^*, t) / \partial x]$ be negative when the operator W is positive for $x \in \Omega$, $0 \leqslant \tau^* \leqslant T$, $0 \leqslant t \leqslant 2\pi$. Then the solution $\alpha_n(t)$ of (4.1) with $\alpha_n(0) = \alpha_n^*(0)$ is asymptotically slow as $\varepsilon \to 0$ for any n for which an isolated solution $\alpha_n^*(\tau^*)$ of the equation $F_n(\alpha_n, \tau^*) = 0$ continuous in τ^* exists.

Proof. It is only necessary to establish the asymptotic stability of the stationary solution $\alpha_n^*(\tau^*)$ of (4.2). The equation for (4.2) in variations can be written in the form

$$\beta_{kn}^{\cdot} = A_{k-l}(\tau^*) \beta_{ln} - ik\beta_{kn} \quad (|k|, \ |l| \leqslant n)$$

or in abridged form as $\beta_n^{\cdot} = A^{(n)}(\tau^*)\beta_n$ where $A^{(n)}(\tau^*) = (A_{k,l}^{(n)}(\tau^*))$ is a matrix-type operator, $A_{k,l}^{(n)}(\tau^*) = A_{k-l,n}(\tau^*) - ik\delta_{kl}$, and $A_{p,n}(\tau^*)$ are the Fourier coefficients for

$$\partial F / \partial x\, (x_n(\tau^*, t), \tau^*, t)$$

Using the condition

$$y^* \operatorname{Re}\left[W \frac{\partial F}{\partial x}(x, \tau^*, t)\right] y \leqslant -\rho y^* y, \quad \rho > 0$$

Substituting

$$y = \sum_{|k| \leqslant n} \beta_{kn} e^{ikt}, \quad x = x_n(t)$$

and integrating over the interval $[0, 2\pi]$ with respect to t, we obtain

$$\sum_{|k|, |l| \leqslant n} \beta_{ln}^* \operatorname{Re}[W A_{k-l, n}] \beta_{kn} \leqslant -\rho \sum_{|k| \leqslant n} |\beta_{kn}|^2 \tag{4.4}$$

Let $W^{(n)} = W \oplus \ldots \oplus W$ $(2n+1)$ times. Then from (4.4) it follows that the operator $\operatorname{Re}[W^{(n)} A^{(n)}(\tau^*)]$ is negative definite and the spectrum of $A^{(n)}(\tau^*)$ lies, by virtue of the Liapunov theorem (see Theorem 5.1 in/6/) in the left half-plane. This means that the solution $\alpha_n^*(\tau^*)$ is asymptotically stable. The uniformity in τ^* of the stability becomes clear when all equations are considered in the Banach space of functions of τ^*, just as in the proof of Theorem 3.

Corollaries. 1°. Let the equation (1.4) be linear: $x^\cdot = A(\tau, t) x + b(\tau, t)$ and let the operator $\operatorname{Re}[WA(\tau, t)]$ be negative for the positive operator W when $0 \leqslant \tau \leqslant T$, $0 \leqslant t \leqslant 2\pi$. Then the solution $\alpha_n(t)$ is asymptotically slow as $\varepsilon \to 0$ for any $n = 0, 1, 2 \ldots$.

2°. Let the equation (1.4) have the form $x^\cdot = A(\tau) x + b(\tau, t)$ and the spectrum of $A(\tau)$ lie in the left half-plane when $0 \leqslant \tau \leqslant T$. Then the solution $\alpha_n(t)$ is asymptotically slow as $\varepsilon \to 0$ for any $n = 1, 2 \ldots$.

5. Example. The above method is illustrated using the example of a one-dimensional nonlinear oscillator with the restoring force possessing a rigid cubic characteristic. The oscillator executes forced oscillations under the action of a harmonic load of slowly increasing frequency and amplitude

$$x^{\cdot\cdot} + \rho x^\cdot + \lambda^2(x + dx^3) = p\cos\varphi, \quad p = p^\circ + \tau, \quad \varphi^\cdot = \omega = \omega^\circ + \tau$$
$$\tau = \begin{cases} 0, & t \leqslant 0 \\ \varepsilon t, & t > 0 \end{cases}$$

Writing the second order equation in the form of a system in $x_1 = x$ and $x_2 = x^\cdot$, we arrive at the system (1.14), (1.15), which assumes the following form at $n = 1$:

$$r_1' = \omega^{-1} r_2 \cos(\psi_2 - \psi_1), \quad \psi_1' = 1 + \omega^{-1} r_1^{-1} r_2 \sin(\psi_2 - \psi_1), \quad r_2' = \omega^{-1}\lambda^2 r_1 (1 + 0{,}75\, dr_1^2) \sin(\psi_2 - \psi_1) - \omega^{-1}\rho r_2 + \omega^{-1} p \cos\psi_2$$
$$\psi_2' = 1 + \omega^{-1} r_2^{-1} \lambda^2 r_1 (1 + 0{,}75\, dr_1^2) \sin(\psi_2 - \psi_1)$$

Here we have

$$x(t) = r_1 \cos(\varphi - \psi_1), \quad x^\cdot(t) = r_2 \cos(\varphi - \psi_2)$$

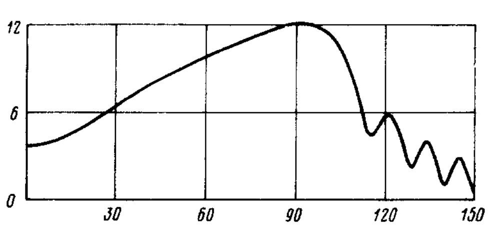

Fig.1

The figure 1 depicts the results of computation (using the Runge-Kutta method) for r_1, obtained for $\lambda = 1$, $d = 0{,}006$, $\omega^\circ = 0{,}9$, $\rho = 0{,}08$, $\varepsilon = 0{,}005$, $p^\circ = 1$ on the interval $0 \leqslant t \leqslant 150$ where the system passes through a fundamental resonance. We see that the amplitude r_1 varies much more slowly than the solution $x(t)$. This means that the system can be solved with any degree of accuracy in relatively few steps, even in the case when ε is not very small and a quasi-stationary approximation cannot be used.

Use of the higher order approximations ($n = 3$ and $n = 5$) gives a noticeable improvement in the accuracy only when the nonlinearity is large and in the case of passage through ultraharmonic resonances.

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